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On $k\beta$-Spaces and Some Other Generalized Metric Spaces.

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1. Introduction

In [11] Wu Lishing introduced the notion of $k\beta$-spaces, which generalizes k-semi-stratifiable spaces due to Lutzer [7]. Recently Xia Shengxiang studied the conditions under which $k\beta$-spaces to be k-semi-stratifiable. We investigate further properties of $k\beta$-spaces, most of which are concerned with the metrization of $k\beta$-spaces. Since the class of k-semi-stratifiable spaces is closely related to that of Nagata spaces, we also investigate the relationship between $k\beta$-spaces and Nagata spaces.

Let (X, τ) be a space and let g be a function from $N\times X$ into τ such that $x\in g(n+1, x)\subset g(n, x)$ for each x in X and n in N. Such a function g is called a COC-function (= countable open covering function). In [3] Hodel introduced some important generalized metric spaces by means of a function COC-function $g: N\times X\to\tau$.

For the definitions of some generalized metric spaces which are not defined in this note, see [1], [2], and [3].

Unless otherwise stated, all topological spaces are assumed to be T_1. The set of positive integers will be denoted by N.

2. Nagata spaces and $k\beta$-spaces.

Instead of giving the original definitions of k-semi-stratifiable spaces [7] and Nagata spaces, we present an equivalent fomulations which are used in this paper. For the actual definitions of these concepts, the reader is referred to [3] and [7].

Definition 2.1 ([11], [12]). (a): A space X is a k-semi-stratifiable space if there is a COC-function g such that $g(n, x_n) \cap K \neq \phi$ for n=1, 2, …, (where K is compact) then the sequence $\langle x_n \rangle$ has a cluster point in K.

(b): A space X is a $k\beta$-space if there is a COC-function g such that $g(n, x_n) \cap K \neq \phi$ for n=1, 2, …, (where K is compact) then the sequence $\langle x_n \rangle$ has a cluster point.

Definition 2.2 ([3]). (a): A space X is a Nagata space if there is a COC-function g such that $g(n, p) \cap g(n, x_n) \neq \phi$ for n=1, 2, …, then p is a cluster point of the sequence $\langle x_n \rangle$.

(b): A space X is a wN-space if there is a COC-function g such that $g(n, p) \cap g(n, x_n) \neq \phi$ for n=1, 2, …, then the sequence $\langle x_n \rangle$ has a cluster point.

Theorem 2.3. Every wN-space is a $k\beta$-space.

A space (X, τ) is called weakly subsequential if each sequence in X which has a cluster point has a subsequence with compact closure. A space X is a $w\sigma$-space if there is a COC-function g such that $p \in g(n, y_n)$, $y_n \in g(n, x_n)$ for n=1, 2, …, then the sequence $\langle x_n \rangle$ has a cluster point, see [2] and [4].

Theorem 2.4. Every weakly subsequential $k\beta$-space is a $w\sigma$-space.

A space X is c-stratifiable if there is a COC-function g such that for each compact set K in X and $p \in X-K$, then there exists n which satisfies $p \notin Cl(g(n,K))$. A space X is called c-Nagata space if it is c-stratifiable and first countable.

Theorem 2.5 A space X is a Nagata space if and only if X is a c-Nagata, $k\beta$-space.

Proof. Let f be a c-Nagata function and g be a $k\beta$-function. Let $h: N\times X \to \tau$ be defined by $h(n,x)=f(n,x)\cap g(n,x)$. Since every first countable k-semi-stratifiable is a Nagata space, it suffices to show that X is k-semi-stratifiable. Let K be a compact subset in X and $h(n, x_n) \cap K\neq \phi \ni y_n$ for $n=1,2,\cdots$. As X is a $k\beta$-space, $\langle x_n \rangle$ has a cluster point p. Since every c-Nagata space is first countable, there is a subsequence $\langle x_{nk}\rangle$ which converges to p.
Let $\{p\} \bigcup \{x_{nk} \mid k=1,2,\cdots\}=C$. Suppose that $p \notin K$. Without loss of generality, we can assume that $K \cap C=\phi$. Since $y_{nk}\in K$, $k=1,2,\cdots$, $\langle y_{nk}\rangle$ has a cluster point q. Since f is a c-Nagata function, there is an m such that $Clf(m,C) \not\ni q$. Then $Clh(m,C) \not\ni q$. Let $V=X-Clh(m,C)$, then there is an i such that $V\ni y_{ni}$, $n_i \geq m$. So we have $y_{ni}\notin h(m, x_{ni}) \supset h(n_i, x_{ni})$ so that $y_{ni}\notin h(n_i, x_{ni})$. This is a contradiction. It follows that X is a k-semi-stratifiable space.

Corollary 2.6 (Lee [5]). A space X is a Nagata space if and only if X is a c-Nagata, wN-space.

A space X is said to have a $G_\delta{}^*$-diagonal if there exists a sequence $\langle \mathcal{G}_n \rangle$ of open covers of X such that, for each $x \in X$, $\bigcap_{n=1}^{\infty} Cl(st(x, \mathcal{G}_n)) = \{x\}$, see [3].

Theorem 2.7. A regular space X is a Nagata space if and only if X is a q, $k\beta$-space with a $G_\delta{}^*$-diagonal.

Proof. Since every regular q-space in which points are G_δ-sets is first countable, let $f: X \times N \to \tau$ be a first countable function. Let $g: N \times X \to \tau$ be a $k\beta$-function. Let $h: N \times X \to \tau$ be defined by $h(n, x) = f(n, x) \cap g(n, x)$. To show that h is a wN-function, let $h(n, p) \cap h(n, x_n) \neq \phi$, for $n = 1, 2, \cdots$. Then there is a sequence $\langle y_n \rangle$ such that $h(n, p) \cap h(n, x_n) \ni y_n$ for all $n \in N$. Since f is a first countable function, the sequence $\langle y_n \rangle$ converges to p. Let $K = \{p\} \cup \{ y_n \mid n = 1, 2, \cdots\}$, then K is compact and $g(n, x_n) \cap K \neq \phi$ for $n = 1, 2, \cdots$. Then $\langle x_n \rangle$ has a cluster point. So h is a wN-function. Note that a regular wN-space with a $G_\delta{}^*$-diagonal is a Nagata space (see, Kotake [5]), whence X is a Nagata space.

A space X is said to have a regular G_δ-diagonal if the diagonal Δ is the intersection of countably many closures of open subsets of $X \times X$ (see [5]).

Theorem 2.8. Every regular $k\beta$-space with a regular G_δ-diagonal is a k-semi-stratifiable space.

3. Metrizability of $k\beta$-spaces.

Theorem 3.1. A space X is metrizable if and only if X is a Hausdorff γ, $k\beta$-space.

Proof. Let f be a γ-function and g be a $k\beta$-function. Let $h: N\times X \to \tau$ be defined by $h(n,x)=f(n,x)\cap g(n,x)$. To show that h is a k-semi-stratifiable function, let K be a compact subset of X and let $h(n, x_n) \cap K\neq \phi$, for n=1, 2, …. As g is a $k\beta$-function, the sequence $\langle x_n \rangle$ has a cluster point p. Since X is a γ-space, X is first countable. Then there is a subsequence $\langle x_{nk}\rangle$ that converges to p. Let $C=\{p\} \cup \{ x_{nk} \mid k=1,2,\cdots\}$. If $p \notin K$, we may assume without loss of generality that $C \cap K=\phi$. Since f is a γ-function, there is an n_0 such that $g(n_0, C) \cap K=\phi$. Now for $n_k \geqq n_0$, $g(n_0, C) \supset g(n_0, x_{nk}) \supset g(n_k, x_{nk})$, so $h(n_k, x_{nk}) \cap K=\phi$. A contradiction. It follows that X is k-semi-stratifiable. Since every k-semi-stratifiable, first countable space is a Nagata space (Lutzer [7]), X is paracompact. In [3], Hodel proved that every β, γ-space is developable. It is well known that every paracompact developable space metrizable, it follows that X is metrizable.

Theorem 3.2. A regular space X is metrizable if and only if X is a $w\theta$, $k\beta$-space with a $G_\delta{}^*$-diagonal.

Corollary 3.3 (Hodel [3]). A regular space X is metrizable if and only if X is a $w\theta$, wN-space with a $G_\delta{}^*$-diagonal.

In [3] Hodel noted that every developable space is a $w\theta$-space.

Therefore, we have the following corollary.

Corollary 3.4. A Hausdorff developable, $k\beta$-space is metrizable

Corollary 3.5 (Hodel [3]). Every Hausdorff developable, wN-space is metrizable.

Since every k-semi-stratifiable space has a $G_\delta{}^*$-diagonal, Theorem 3.2 generalizes the following result of Martin.

Corollary 3.6 (Martin [10]). A regular space X is metrizable if and only if X is a k-semi-stratifiable, quasi-γ-space.

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